JOURNAL OF APPROXIMATION THEORY **90**, 1–8 (1997)
ARTICLE NO. AT963075

Balázs–Shepard Operators on Infinite Intervals, II

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Communicated by Doron S. Lubinsky

Received December 11, 1995; accepted in revised form August 2, 1996

The weighted uniform convergence of the Balázs–Shepard operator is considered on the real line. As a consequence of the main result, it is proved that for a wide class of weights, rational functions are always dense in the space of continuous functions, in contrast to the polynomials where the Akhiezer–Babenko condition is necessary for such density.

In [1] the convergence of the so-called Balázs–Shepard operators

$$S_n(f, x) := \frac{\sum_{k=-n}^{n} f(x_k)(x-x_k)^{-2}}{\sum_{k=-n}^{n} (x-x_k)^{-2}} \tag{1}$$

was considered on $\mathbf{R}$. Here

$$x_k := \frac{\lambda_n k}{n} \qquad (k=0, \pm 1, ..., \pm n) \tag{2}$$

are equidistant nodes where $\lambda_n > 0$ is a real number depending on n. In the case where $f(x)$ has equal finite limits at $\pm\infty$ the error estimates obtained in [1] were quite satisfactory, but when $f(x)$ is unbounded at $\pm\infty$ we could not get results for the original operator, only for some modification. The purpose of this paper is to settle the problem of weighted approximation by the Balázs–Shepard operators.

To do so we must define our weight function $w(x) = e^{-Q(x)}$ by the following properties: There exists an $a \geqslant 0$ such that

(i) $Q(x)$ is even, $\lim_{x\to\infty} Q(x) = \infty$;

* Research of this author was supported by the Hungarian National Science Foundation under Grant T017425.

0021-9045/97 $25.00

(ii) $Q'(x) \geqslant 0$ is either strictly monotone increasing or bounded in $[a, \infty)$; and

(iii) in the case where $Q'(x)$ is strictly monotone increasing in $[a, \infty)$, $Q'(x+1) \leqslant AQ'(x)$ ($x \in [a, \infty)$, $A > 0$ independent of x).

These conditions are almost the same as those in Ditzian–Totik [2, p. 181], except that we allow $Q'(x)$ to be bounded and (ii) and (iii) are required only for sufficiently large x, thus considering a wider class of weights. For example, weights like

$$Q(x) = \log^{\beta}(1+|x|), \qquad Q(x) = |x|^{\beta} \quad (\beta > 0), \qquad Q(x) = e^{c\,|x|} \quad (c > 0)$$

are included, but $Q(x) = e^{x^2}$ is excluded by (iii).

In connection with these weights, for each $t \in [a, \infty)$ we define t^* as

(a) the unique solution of $tQ'(t^*) = 1$ if $Q'(x) \uparrow \infty$ as $x \uparrow \infty$, and

(b) ∞ if $Q'(x)$ is bounded as $x \to \infty$.

Again, (a) is taken from [2, p. 181]. After these preliminaries, the modulus of continuity is defined as (compare [2, 11.2.6])

$$\Omega(f, t)_w := \sup_{0 < h \leqslant t} \|w(x)(f(x+h/2) - f(x-h/2))\|_{L_\infty[-h^*, h^*]} \tag{3}$$

for any $f \in C(\mathbf{R})$ satisfying

$$\lim_{|x| \to \infty} w(x) f(x) = 0. \tag{4}$$

In order to formulate our result, we need another characteristic of the function $f(x)$ which measures the rate of convergence in (4) (cf. also [1]):

$$\varepsilon(f, y) := \sup_{|x| \geqslant y} w(x)\,|f(x)| \qquad (y \geqslant 0). \tag{5}$$

This is a monotone decreasing function of y, and for functions satisfying (4) evidently

$$\lim_{y \to \infty} \varepsilon(f, y) = 0. \tag{6}$$

THEOREM. *For any $f(x)$ satisfying* (4) *we have*

$$\|w(x)(f(x) - S_n(f, x))\|_{\mathbf{R}}$$
$$= O\left(\Omega\left(f, \frac{\lambda_n \log n}{n}\right)_w + \varepsilon(f, \mu_n) + \frac{w(\lambda_n)}{w(\mu_n)} + \frac{\lambda_n^2}{n w(\lambda_n)}\right) \tag{7}$$

for any $\lambda_n > \mu_n > 0$.

Before proving this statement, we formulate an important

COROLLARY. *For any* $f(x)$ *satisfying* (4), *there exist* $\lambda_n > \mu_n > 0$ *such that*

$$\lim_{n\to\infty} \|w(x)(f(x) - S_n(f, x))\|_{\mathbf{R}} = 0. \tag{8}$$

Proof of the Corollary. Let λ_n be such that

$$w(\lambda_n) = \frac{\lambda_n^2}{n^{1/2}}.$$

(This is, by no means, an optimal choice of λ_n, but it suffices to prove the corollary. As for the optimal choice, it depends on the structural properties of the function; see the examples at the end of the paper.) It follows from (i) above that such a λ_n (uniquely) exists; in fact $\lambda_n \in (0, n^{1/4})$ and $\lim_{n\to\infty} \lambda_n = \infty$. Now choose μ_n such that $Q(\mu_n) = \frac{1}{2} Q(\lambda_n)$; evidently $\lim_{n\to\infty} \mu_n = \infty$. With this choice of λ_n, μ_n the right-hand side of (7) takes the form

$$O\left(\Omega\left(f, \frac{\log n}{n^{3/4}}\right)_w + \varepsilon(f, \mu_n) + w(\lambda_n)^{1/2} + \frac{1}{n^{1/2}}\right)$$

which tends to 0 as $n \to \infty$ if we take into account the relation (9) of the lemma to be proved below.

In order to prove our theorem we need some basic properties of the modulus of continuity defined above. These properties are not mentioned in Ditzian–Totik [2] (moreover, the class of weight functions considered here are more general); therefore for completeness we provide a proof.

LEMMA. *For any* $f(x)$ *satisfying* (4) *we have*

$$\lim_{t=0+} \Omega(f, t)_w = 0 \tag{9}$$

and

$$\Omega(f, \lambda t)_w \leqslant c(w)(\lambda + 1)\, \Omega(f, t)_w \qquad (\lambda t \leqslant 2), \tag{10}$$

where $c(w) > 0$ *depends only on* w.

Proof. We have

$$\begin{aligned} &w(x)\, |f(x+h/2) - f(x-h/2)| \\ &\quad \leqslant |w(x+h/2)\, f(x+h/2) - w(x-h/2)\, f(x-h/2)| \\ &\qquad + |f(x+h/2)|\, [w(x) - w(x+h/2)] + |f(x-h/2)|\, [w(x-h/2) - w(x)]. \end{aligned}$$

Evidently, we may assume that here $a \leqslant |x| \leqslant h^*$. By the mean value theorem we get for the above quantity

$$= o(1) + w(x+h/2)\,|f(x+h/2)|\,[e^{Q(x+h/2)-Q(x)} - 1]$$
$$+ w(x-h/2)\,|f(x-h/2)|\,[e^{Q(x)-Q(x-h/2)} - 1]$$
$$= o(1) + w(x+h/2)\,|f(x+h/2)|\,[e^{hQ'(\xi)/2} - 1]$$
$$+ w(x-h/2)\,|f(x-h/2)|\,[e^{hQ'(\eta)/2} - 1]$$
$$(x-h/2 < \eta < x < \xi < x+h/2), \tag{11}$$

where $o(1)$ refers to $h \to 0$. Here in the case where Q' is bounded the right-hand side goes to 0 as $h \to 0$. Otherwise we have by (iii) and (a) above for $0 < h < 2$ and $a \leqslant |x| \leqslant h^*$

$$hQ'(\eta) \leqslant hQ'(\xi) \leqslant hQ'(x+h/2) \leqslant hQ'(x+1)$$
$$\leqslant AhQ'(x) \leqslant AhQ'(h^*) = e^A.$$

Thus given an arbitrary $\varepsilon > 0$, for $|x| \geqslant x_1(\varepsilon)$, $w(x \pm h/2)\,|f(x \pm h/2)| < \varepsilon$, while for $|x| < x_1(\varepsilon)$

$$e^{hQ'(\eta)/2} - 1 \leqslant e^{hQ'(\xi)/2} - 1 \leqslant e^{O(h)} - 1 < \varepsilon \qquad \text{if} \quad h \leqslant h_0(\varepsilon)$$

and thus in both cases the right-hand side of (11) goes to 0 as $h \to 0$.

In order to prove (10), first we note that it is sufficient to prove it for $\lambda \geqslant A$, since if $\lambda < A$, then by the monotonicity of Ω we get

$$\Omega(f, \lambda t)_w < \Omega(f, At)_w \leqslant c_1(w)(A+1)\,\Omega(f, t)_w$$
$$\leqslant c_1(w)(A+1)(\lambda+1)\,\Omega(f, t)_w$$

which proves (10) with $c(w) = c_1(w)(A+1)$. Thus let $\lambda > A$, and evidently, we may assume that λ is an integer. We obtain from (3) with the substitution $h = \lambda H$

$$\Omega(f, \lambda t)_w = \sup_{0 < H \leqslant t} \|w(x)(f(x+\lambda H/2) - f(x-\lambda H/2))\|_{L_\infty[-(\lambda H)^*, (\lambda H)^*]}$$
$$\leqslant \sup_{0 < H \leqslant t} \sum_{j=1-\lambda}^{\lambda-1} \|w(x+jH/2)[f(x+(j+1)H/2)$$
$$- f(x+(j-1)H/2)]\|_{L_\infty[-(\lambda H)^*, (\lambda H)^*]}$$
$$\cdot \left\| \frac{w(x)}{w(x+jH/2)} \right\|_{L_\infty[-(\lambda H)^*, (\lambda H)^*]}. \tag{12}$$

In Case (a) we have by $\lambda H \leqslant \lambda t \leqslant x_0$ and (iii)

$$Q'((\lambda H)^* + \lambda H/2) \leqslant AQ'((\lambda H)^*) = \frac{A}{\lambda H} \leqslant \frac{1}{H} = Q'(H^*),$$

whence by the monotonicity of Q' we obtain

$$|x + jH/2| \leqslant |x| + \lambda H/2 \leqslant (\lambda H)^* + \lambda H/2 \leqslant H^* \quad (|x| \leqslant (\lambda H)^*,\ j = -\lambda, ..., \lambda).$$

Thus we obtain from (12) (extending the estimates trivially for $|x| \leqslant a$)

$$\begin{aligned} \Omega(f, \lambda t)_w = &\sup_{0 < H \leqslant t} \sum_{j=1-\lambda}^{\lambda-1} \|w(x)[f(x+H/2) - f(x-H/2)]\|_{L_\infty[-H^*, H^*]} \\ &\cdot \left\| \frac{w(x)}{w(x+jH/2)} \right\|_{L_\infty[-(\lambda H)^*, (\lambda H)^*]} \\ &\leqslant \Omega(f, t)_w\,(2\lambda - 1) \left\| \frac{w(x)}{w(|x| + \lambda H/2)} \right\|_{L_\infty[-(\lambda H)^*, (\lambda H)^*]}, \end{aligned} \tag{13}$$

and these steps are correct also in Case (b), since then the norms are to be taken over the entire real line. Here the last norm is evidently bounded for $|x| \leqslant x_0$; otherwise

$$\frac{w(x)}{w(x+\lambda H/2)} = e^{Q(|x| + \lambda H/2) - Q(x)} = e^{\lambda H Q'(\xi)/2} \qquad (a \leqslant |x| < \xi < |x| + \lambda H/2),$$

again by the mean value theorem. Here in Case (a) we get

$$e^{\lambda H Q'(\xi)} \leqslant e^{\lambda H Q'(|x| + \lambda H/2)} \leqslant e^{A\lambda H Q'(|x|)} \leqslant e^{A\lambda H Q'((\lambda H)^*)} = e^A$$

$$(a \leqslant |x| \leqslant (\lambda H)^*),$$

while in Case (b) $e^{\lambda H Q'(\xi)} \leqslant e^{\|Q'\|_{L_\infty(-\infty, \infty)}}$. These estimates together with (13) completely prove (10).

Proof of the Theorem. By symmetry, it is sufficient to prove for $x \geqslant 0$. We distinguish two cases.

Case 1. $0 \leqslant x \leqslant \lambda_n$. Let

$$|x - x_j| := \min_{|k| \leqslant n} |x - x_k| \leqslant \frac{\lambda_n}{2n}$$

and

$$K_n(x) := \left\{ k\colon k \in \mathbf{N},\ |k| \leqslant n,\ |x - x_k| \leqslant \min\left(\frac{2}{Q'(|x + x_k|/2)}, 1 \right) \right\}.$$

First we estimate the quantities

$$B_k := \frac{w(x)}{w(|x + x_k|/2)} \qquad (k \in K_n(x)).$$

If $x \leqslant a + \frac{1}{2}$ then B is evidently bounded. Now if $x > a + \frac{1}{2}$ and $x_k \leqslant x$ then $B_k \leqslant 1$. Thus we may assume that $a + \frac{1}{2} < x < x_k$, whence by the mean value theorem

$$B_k = e^{Q(|x + x_k|/2) - Q(x)} = e^{((x_k - x)/2)\, Q'(\xi)} \qquad \left(\xi \in \left(x, \frac{|x + x_k|}{2} \right) \right).$$

In Case (b), this is bounded by $e^{(1/2)\, \|Q'\|_{L_\infty(-\infty, \infty)}}$. In Case (a), by the monotonicity of Q' we get

$$B_k \leqslant e^{((x_k - x)/2)\ Q'((x + x_k)/2)} \leqslant e.$$

Hence $B_k \leqslant B$ in all cases considered.

Now if $k \in K_n(x)$ then evidently $|x + x_k|/2 \leqslant (|x - x_k|/2)^*$, and thus by the definition of the modulus of continuity in (3) and (10) we get, using the previous estimate,

$$\begin{aligned}
&\frac{w(x)}{\sum_{k=-n}^{n} (x - x_k)^{-2}} \sum_{k \in K_n(x)} \frac{|f(x) - f(x_k)|}{(x - x_k)^2} \\
&\quad \leqslant \max_{k \in K_n(x)} \frac{w(x)}{w(|x + x_k|/2)} (x - x_j)^2 \sum_{|k| \leqslant n} \frac{\Omega(|x - x_k|)_w}{(x - x_k)^2} \\
&\quad \leqslant B(x - x_j)^2\, \Omega\left(\frac{\lambda_n \log n}{n} \right)_w \left[\frac{n}{\lambda_n \log n} \sum_{|k| \leqslant n} \frac{1}{|x - x_k|} + \sum_{|k| \leqslant n} \frac{1}{(x - x_k)^2} \right] \\
&\quad \leqslant \frac{\lambda_n^2}{n^2}\, \Omega\left(\frac{\lambda_n \log n}{n} \right)_w O\left[\frac{n}{\lambda_n \log n} \cdot \frac{n \log n}{\lambda_n} + \frac{n^2}{\lambda_n^2} \right] \\
&\quad = O\left(\Omega\left(\frac{\lambda_n \log n}{n} \right)_w \right).
\end{aligned}$$

On the other hand, we obtain

$$\frac{w(x)}{\sum_{k=-n}^{n}(x-x_k)^{-2}} \sum_{\min(1,\, 2/Q'(|x+x_k|/2))<|x-x_k|} \frac{|f(x)-f(x_k)|}{(x-x_k)^2}$$
$$\leqslant \frac{w(x)\,\lambda_n^2}{2n^2 w(\lambda_n)} \sum_{|k|\leqslant n} Q'\left(\frac{|x+x_k|}{2}\right)^2.$$

Here, if Q' is bounded, we get the last term on the right-hand side of (7). Otherwise, by (a)

$$w(x)\,Q'\left(\frac{|x+x_k|^2}{2}\right) \leqslant e^{2\log Q'(x+1)-Q(x)} \leqslant e^{2\log(A^x Q'(1))-\int_{x/2}^{x} Q'(t)\,dt}$$
$$= O(e^{2x\log A-(x/2)\,Q'(x/2)}) = O(1),$$

whence again we get the same estimate.

Collecting all of these estimates we get

$$w(x)\,|f(x)-S_n(f,x)| = O\left(\Omega\left(\frac{\lambda_n \log n}{n}\right)_w + \frac{\lambda_n^2}{n w(\lambda_n)}\right) \qquad (|x|\leqslant\lambda_n).$$

Case 2. $x>\lambda_n$. Then

$$w(x)\,|f(x)-S_n(f,x)|$$
$$\leqslant \frac{\sum_{|k|\leqslant n} w(x)[\,|f(x)|+|f(x_k)|\,]/(x-x_k)^2}{\sum_{|k|\leqslant n}(x-x_k)^{-2}}$$
$$\leqslant \varepsilon(f,\lambda_n)+w(\lambda_n)\left[\frac{\left(\begin{array}{r}\sum_{|k|\leqslant n\mu_n/\lambda_n} O(w(x_k)^{-1})/(x-x_k)^2\\ +\sum_{n\mu_n/\lambda_n<|k|\leqslant n}\varepsilon(f,\mu_n)/(w(\lambda_n)(x-x_k)^2)\end{array}\right)}{\sum_{|k|\leqslant n}(x-x_k)^{-2}}\right]$$
$$\leqslant \varepsilon(f,\lambda_n)+O\left(\frac{w(\lambda_n)}{w(\mu_n)}+\varepsilon(f,\mu_n)\right)$$
$$= O\left(\frac{w(\lambda_n)}{w(\mu_n)}+\varepsilon(f,\mu_n)\right) \qquad (|x|>\lambda_n).$$

Hence the proof of the theorem is complete.

Examples. Let us consider the special weights listed at the beginning. In all these cases we assume, for simplicity, that

$$\Omega(f,t)_w = O(t^\alpha) \qquad (0<\alpha<1).$$

Now if $Q(x)=\log^{\beta}(1+|x|)$ $(\beta>0)$ then for the function class $|f(x)|=O(|x|^{\gamma})$ $(|x|\to\infty,\ 0<\gamma<\beta)$ and with the choice

$$\lambda_n=(n^{1-\alpha}\log^{\alpha} n)^{1/(2+\beta-\alpha)}, \qquad \mu_n=\lambda_n^{\beta/(2\beta-\gamma)}$$

the right-hand side of (7) takes the form

$$O\left(\left(\frac{\log^{2+\beta} n}{n^{1+\beta}}\right)^{\alpha/(2+\beta-\alpha)}+(n^{1-\alpha}\log^{\alpha} n)^{-\beta(\beta-\gamma)/((2\beta-\gamma)(2+\beta-\gamma))}\right).$$

Next, if $Q(x)=|x|^{\beta}$ $(\beta>0)$, then for the class of functions $|f(x)|=O(e^{c\,|x|^{\beta}})$ $(0<c<1)$, with $\lambda_n=\log^{1/\beta} n$, $\mu_n=\lambda_n/(2-c)^{1/\beta}$ we obtain the error estimate

$$O\left(\left(\frac{\log^{1/\beta} n}{n}\right)^{\alpha}\right).$$

Finally, if $Q(x)=e^{|x|}$, then for $|f(x)|=O(e^{ce^{|x|}})$ $(0<c<1)$ with the choice $\lambda_n=(\log\log n)^{1/\beta}$, $\mu_n=(\lambda_n^{\beta}-\log(2-c))^{1/\beta}$ we get

$$O\left(\left(\frac{\log n(\log\log n)^{1/\beta}}{n}\right)^{\alpha}\right).$$

Finally, we mention that our corollary above can be interpreted such that the *rational* functions are always dense in the space of continuous functions with respect to the weights $e^{-Q(x)}$ considered in our theorem. This is in sharp contrast with the polynomial approximation where the density condition is

$$\int_{-\infty}^{\infty}\frac{Q(x)}{1+x^2}\,dx=\infty.$$

As we have seen, no such condition is necessary for rational functions, for a wide class of weights.

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